NEW FORMAT

AJ Sadler

Mathematics Specialist

Student Book

Units 1 & 2

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PREFACE

This text targets units one and two of the West Australian course *Mathematics Specialist*. Chapters one to eight cover the content of Unit One and chapters nine to thirteen cover Unit Two.

The West Australian course, *Mathematics Specialist*, is based on the Australian Curriculum Senior Secondary course *Specialist Mathematics*. At the time of writing there are only small differences between the content for units one and two of the West Australian course and units one and two of the Australian Curriculum course, so this text would also be suitable for anyone following units one and two of the Australian Curriculum course, *Specialist Mathematics*. The main differences are that the West Australian units include vertical translations of trigonometric graphs, solving linear equations in two variables by matrix methods, and when considering converse and contrapositive statements the inverse is also covered.

The book contains text, examples and exercises containing many carefully graded questions. A student who studies the appropriate text and relevant examples should make good progress with the exercise that follows.

The book commences with a section entitled **Preliminary work**. This section briefly outlines work of particular relevance to this unit that students should either already have some familiarity with from the mathematics studied in earlier years, or for which the brief outline included in the section may be sufficient to bring the understanding of the concept up to the necessary level.

As students progress through the book they will encounter questions involving this preliminary work in the **Miscellaneous Exercises** that feature at the end of each chapter. These miscellaneous exercises also include questions involving work from preceding chapters to encourage the continual revision needed throughout the unit.

Some chapters commence with a '**Situation**' or two for students to consider, either individually or as a group. In this way students are encouraged to think and discuss a situation, which they are able to tackle using their existing knowledge, but which acts as a fore-runner and stimulus for the ideas that follow. Students should be encouraged to discuss their solutions and answers to these situations and perhaps to present their method of solution to others. For this reason answers to these situations are generally not included in the book.

At times in this series of texts I have found it appropriate to perhaps go a little outside the confines of the syllabus for the unit involved. In this regard readers will find that in unit two I consider

$$\sin P \pm \sin Q \text{ and } \cos P \pm \cos Q,$$

and when considering matrix equations of the form $AX = B$, my considerations go beyond X and B being column matrices. I also include shear transformations and mention proof by exhaustion.

Alan Sadler

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IMPORTANT NOTE

This text has been written based on my interpretation of the appropriate Mathematics Specialist syllabus documents as they stand at the time of writing. It is likely that as time progresses some points of interpretation will become clarified and perhaps even some changes could be made to the original syllabus. I urge teachers of the Mathematics Specialist course, and students following the course, to check with the appropriate curriculum authority to make themselves aware of the latest version of the syllabus current at the time they are studying the course.

Acknowledgements

As with all of my previous books I am again indebted to my wife, Rosemary, for her assistance, encouragement and help at every stage.

To my three beautiful daughters, Rosalyn, Jennifer and Donelle, thank you for the continued understanding you show when I am 'still doing sums' and for the love and belief you show in me.

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Mathematics Specialist

Unit One

UNIT ONE PRELIMINARY WORK

This book assumes that you are already familiar with a number of mathematical ideas from your mathematical studies in earlier years.

This section outlines the ideas which are of particular relevance to Unit One of the Mathematics Specialist course and for which familiarity will be assumed, or for which the brief explanation given here may be sufficient to bring your understanding of the concept up to the necessary level.

Read this 'Preliminary work' section and if anything is not familiar to you, and you don't understand the brief mention or explanation given here, you may need to do some further reading to bring your understanding of the concepts up to an appropriate level for this unit. (If you do understand the work but feel somewhat 'rusty' with regards to applying the ideas, some of the chapters afford further opportunities for revision, as do some of the questions in the miscellaneous exercises at the end of chapters.)

- Chapters in this book will continue some of the topics from this preliminary work by building on the assumed familiarity with the work.
- The **miscellaneous exercises** that feature at the end of each chapter may include questions requiring an understanding of the topics briefly explained here.
- Familiarity with some of the content of unit one of the *Mathematics Methods* course will be useful for this unit of *Mathematics Specialist*. It is anticipated that students will be studying that unit at the same time as this one and that the useful familiarity with content will be reached in time for its use in this unit.

Number

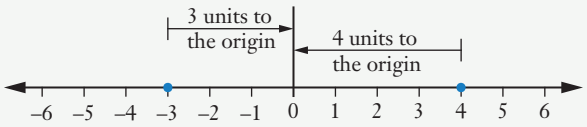
The set of numbers that you are currently familiar with is called the set of **real numbers**. We use the symbol $\mathbb{R}$ for this set. $\mathbb{R}$ contains many subsets of numbers such as the whole numbers, the integers, the rational numbers, the irrational numbers etc.

The absolute value

A concept that you may have met before is that of the absolute value of a number. This is the distance on the number line that the number is from the origin.

The absolute value of -3, written $|-3|$, is 3 because -3 is three units from the origin.

The absolute value of 4, written $|4|$, is 4 because 4 is four units from the origin.



The absolute value makes no distinction between numbers that are to the left of the origin and those that are to the right of the origin.

Thus $|3| = 3$ $|-3| = 3$ $|5| = 5$ $|-5| = 5$ $|3.1| = 3.1$ $|-3.1| = 3.1$ $|8.2| = 8.2$ $|-8.2| = 8.2$

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Trigonometry

It is assumed that you are already familiar with the use of the theorem of Pythagoras and the ideas of sine, cosine and tangent to determine the unknown sides and angles in right triangles. Your answers should be given to the accuracy requested or, if none is specifically requested, to an accuracy that is appropriate for the accuracy of the given data and the nature of the situation.

An understanding of the concept of a bearing being measured clockwise from North, e.g. 080°, and using compass points, e.g. N 80° E, is also assumed.

With $\text{sine} = \frac{\text{opp}}{\text{hyp}}$, $\text{cosine} = \frac{\text{adj}}{\text{hyp}}$ and $\text{tangent} = \frac{\text{opp}}{\text{adj}}$, it follows that

$$\tan A = \frac{\sin A}{\cos A}$$

This is a result you may not be familiar with but that we will use in this unit.

You may be familiar with:

- the formula $A = \frac{1}{2}ab \sin C$ to determine the area of a triangle
- the sine rule $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$
- the cosine rule $c^2 = a^2 + b^2 - 2ab \cos C$
- the use of the unit circle to give meaning to the sine and cosine of angles bigger than 90°
- expressing the trigonometric ratios of some angles as exact values,

or you will become familiar with these concepts as your concurrent study of unit one of the *Mathematics Methods* course progresses.

It is also assumed that you are familiar with the idea that *similar triangles* have corresponding sides in the same ratio – the underlying idea on which the trigonometrical ratios of right triangles are based.

Use of algebra

It is assumed that you are already familiar with manipulating algebraic expressions, in particular:

- Expanding and simplifying:

For example	$4(x + 3) - 3(x + 2)$	expands to	$4x + 12 - 3x - 6$
		which simplifies to	$x + 6$
	$(x - 7)(x + 1)$	expands to	$x^2 + 1x - 7x - 7$
		which simplifies to	$x^2 - 6x - 7$
	$(2x - 7)^2$, i.e. $(2x - 7)(2x - 7)$	expands to	$4x^2 - 28x + 49$

- Factorising:

For example,	$21x + 7$	factorises to	$7(3x + 1)$
	$x^2 - 6x - 7$	factorises to	$(x - 7)(x + 1)$
	$x^2 - y^2$	factorises to	$(x - y)(x + y)$.

  the last one being referred to as the *difference of two squares*.

- Solving equations: In particular linear, simultaneous and quadratic equations.

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Similar triangles

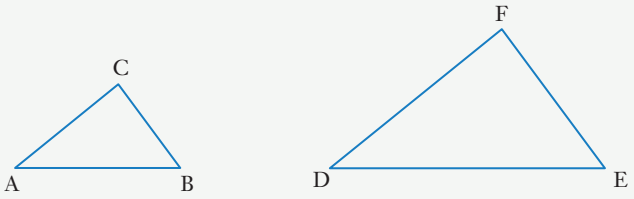
Whilst it is assumed that you are familiar with the concept of two triangles being similar (as mentioned at the end of the trigonometry section of this *Preliminary work* section) it is further assumed that you can apply these ideas in the formulation of a proof. Read through the following by way of a reminder of these ideas.

If two triangles are similar, corresponding sides are in the same ratio. (One triangle is like a 'photographic enlargement' of the other.)

For example, if the two triangles ABC and DEF, shown below, are similar then

$$AB : DE = AC : DF = BC : EF$$

i.e.
$$\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$$



Note If two triangles are similar, the corresponding angles are equal.

i.e., in the similar triangles shown above

$\angle CAB = \angle FDE$,

$\angle ABC = \angle DEF$,

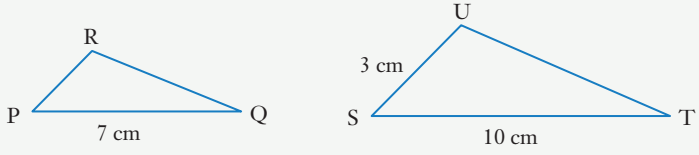
$\angle ACB = \angle DFE$.

We can use the symbol '~' to indicate that two shapes are similar.

Thus $\triangle ABC \sim \triangle DEF$

The order of the letters *is* significant and indicates corresponding sides and angles. Thus, for triangles ABC and DEF above, it would **not** be correct to say $\triangle ABC \sim \triangle FED$.

Knowing that two triangles are similar can sometimes allow us to determine the length of some of the sides. For example if we were told that for the triangles sketched below,



$\triangle PQR \sim \triangle STU$

Then $\frac{PQ}{ST} = \frac{PR}{SU}$

i.e. $\frac{7}{10} = \frac{x}{3}$ where x cm is the length of PR.

Solving gives $x = 2.1$. Thus PR is of length 2.1 cm.

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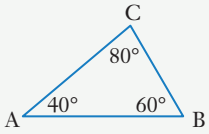
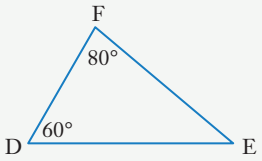
To determine whether two triangles are similar we can:

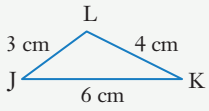
See if the three angles of one triangle are equal to the three angles of the other triangle.

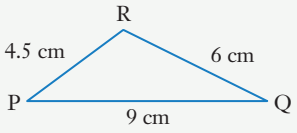




△ABC and △EDF are similar. Reason: Corresponding angles equal.

Or: See if the lengths of corresponding sides are in the same ratio.





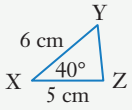
$JK : PQ = 6 : 9$
$= 2 : 3$

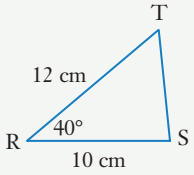
$KL : QR = 4 : 6$
$= 2 : 3$

$JL : PR = 3 : 4.5$
$= 2 : 3$

△JKL and △PQR are similar. Reason: Corresponding sides in same ratio.

Or: See if the lengths of two pairs of corresponding sides are in the same ratio and the angles between the sides are equal.





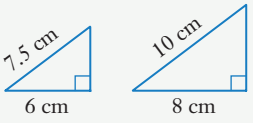
$XZ : RS = 5 : 10$
$= 1 : 2$

$XY : RT = 6 : 12$
$= 1 : 2$

The angle between XY and XZ = the angle between RT and RS.

△XZY and △RST are similar. Reason: Two pairs of corresponding sides in same ratio and the *included* angles equal.

Note: If the two triangles are right angled, the corresponding sides that are in the same ratio need not *include* the right angle.



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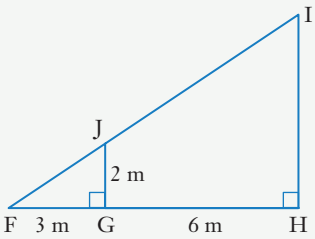
Example

Given that in the diagram shown

FG = 3 m, GH = 6 m, JG = 2 m.

prove that IH is of length 6 m.



Solution

Given: Diagram as shown.

To prove: IH = 6 m.

Proof: In triangles FGJ and FHI:

$\angle FGJ = \angle FHI$ (Each angle = 90°.)

$\angle JFG = \angle IFH$ (Same angle.)

$\therefore \quad \angle FJG = \angle FIH$ (Third angle in each triangle.)

Thus $\triangle FGJ \sim \triangle FHI$ (Corresponding angles equal.)

$\therefore \quad \dfrac{JG}{IH} = \dfrac{FG}{FH}$ i.e., if IH is x m in length, $\dfrac{2}{x} = \dfrac{3}{9}$

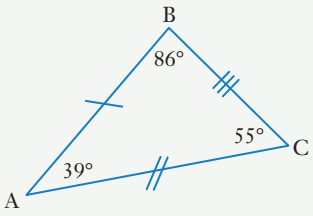
$x = 6$

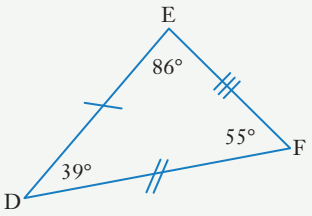
Thus IH is of length 6 m, as required.

Notice how known truths are referred to in order to justify statements. Such justifications do not need to be 'essays' but are instead a brief statement to clearly indicate which truth justifies the statement.

Congruent triangles

If two triangles are identical, i.e. *congruent*, the three angles and three sides of one will match the three angles and three sides of the other. For example, the triangles ABC and DEF shown below are congruent.





We write: $\triangle ABC \cong \triangle DEF$

However, we do not need to know that all six pieces of information,

A A A S S S,

'match' before being able to say that two triangles are congruent.

In fact, to prove two triangles congruent, we need only to show one of the sets of facts shown on the next page.

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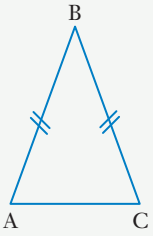
To prove two triangles congruent we need only show one of the following:

SSS	The lengths of the three sides of one triangle are the same as the lengths of the three sides of the other triangle.
SAS	The lengths of two sides of one triangle are the same as the lengths of two sides of the other triangle and the included angles are equal.
AA corresponding S	Two angles of one triangle equal two angles of the other and one side in one triangle is the same length as the corresponding side in the other.
RHS	Both triangles are right angled, the two hypotenuses are of the same length and one other side from each triangle are of equal length.

Example

If we define an isosceles triangle to be a triangle that has two of its sides of the same length, prove that two of its angles must also be equal.



Solution

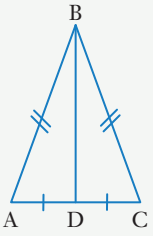
Given: $\triangle ABC$ with AB = CB.

To prove: $\angle BAC = \angle BCA$

Construction: Draw the line from B to D, the midpoint of AC.
(See second diagram on the right.)

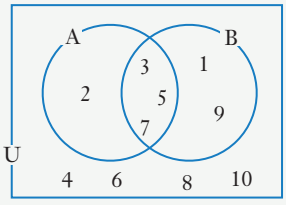
Proof: In triangles ADB and CDB

	AB = CB	(Given.)
	AD = CD	(D is the midpoint of AC.)
	BD = BD	(Common to both triangles.)
$\therefore$	$\triangle ADB \cong \triangle CDB$	(SSS)
Hence	$\angle BAD = \angle BCD$	(Corresponding angles)
and so	$\angle BAC = \angle BCA$ as required.	



Sets and Venn diagrams

Some familiarity with sets and **Venn diagrams** is assumed. The **Venn diagram** on the right shows the **universal set**, U, which contains all of the **elements** currently under consideration, and the sets A and B contained within it.



We use 'curly brackets' to list a set. Thus for the sets shown

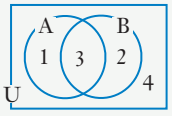
$U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{2, 3, 5, 7\}$ and $B = \{1, 3, 5, 7, 9\}$.

Using the symbol $\in$ for 'is a member of ': $3 \in A$, $9 \in B$, $9 \notin A$.

For the union of two sets we use '$\cup$', for the intersection of two sets we use '$\cap$', for the number of elements in a set we use n(A) or $|A|$ and for the complement of a set we use A′ or $\overline{A}$. Thus for the sets A and B shown above

$A \cap B = \{3, 5, 7\}$, $A \cup B = \{1, 2, 3, 5, 7, 9\}$,
$n(A) = 4$ and $A' = \{1, 4, 6, 8, 9, 10\}$.

Sometimes we use the Venn diagram to show the number of elements in the particular regions, rather than the elements themselves.



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